NEW EXAMPLES OF HYPERBOLIC OCTIC SURFACES IN $\mathbb{P}^3$

BERNARD SHIFFMAN AND MIKHAIL ZAIDENBERG

Abstract. We show that a general small deformation of the union of two general cones in $\mathbb{P}^3$ of degree ≥ 4 is Kobayashi hyperbolic. Hence we obtain new examples of hyperbolic surfaces in $\mathbb{P}^3$ of any given degree $d \geq 8$.

It was shown by Clemens [2] that a very general surface X_d of degree $d \geq 5$ in $\mathbb{P}^3$ has no rational curves; G. Xu [11] showed that X_d also has no elliptic curves (and in fact has no curves of genus ≤ 2), i.e. X_d is algebraically hyperbolic. According to the Kobayashi Conjecture, X_d must even be Kobayashi hyperbolic, and hence does not possess non-constant entire curves $\mathbb{C} \to X_d$. The latter property is known to be open in the Hausdorff topology on the projective space of degree d surfaces [12], and it does hold for a very general surface of degree at least 15 [3, 5, 7].

Examples of hyperbolic surfaces in $\mathbb{P}^3$ have been given by many authors; see the references in our previous papers [9, 10], where more examples are given. So far, the minimal degree of known examples is 8; the first family of examples of degree 8 hyperbolic surfaces in $\mathbb{P}^3$ was found by Fujimoto [6] and independently by Duval [4]. In [10], we introduced a deformation method, which we used to construct a new degree 8 hyperbolic surface. In this note, we use a simple form of our deformation method to construct another degree 8 example, which is a deformation of the union of two quartic cones. Actually, our construction provides examples in any degree $d \geq 8$.

It follows from an observation by Mumford and Bogomolov, proved in [8], that every surface in $\mathbb{P}^3$ of degree at most 4 contains rational or elliptic curves. However, it remains unknown whether there exist hyperbolic surfaces in $\mathbb{P}^3$ in the remaining degrees $d = 5, 6, 7$.

To describe our examples, we consider an algebraic curve C in a plane $H \subset \mathbb{P}^3$. We let $\langle C, p\rangle$ denote the cone formed by the union of lines through a fixed point $p \in \mathbb{P}^3 \setminus H$ and points of C. By a cone in $\mathbb{P}^3$, we mean a cone of the form $X = \langle C, p\rangle$. If $C' = X \cap H'$, where H' is an arbitrary plane not passing through p, then we also have $X = \langle C', p\rangle$. We observe that $\deg X = \deg C$.

Theorem. *For $m, n \geq 4$, a general small deformation of the union $X = X' \cup X''$ of two general cones in $\mathbb{P}^3$ of degrees m and n, respectively, is a hyperbolic surface of degree $m+n$.*

Proof. Let $X = X' \cup X'' \subset \mathbb{P}^3$ be the union of two general cones of respective degrees $m, n \geq 4$. We choose coordinates $(z_1 : z_2 : z_3 : z_4) \in \mathbb{P}^3$ so that X', X'' are cones through the points $a = (0 : 0 : 0 : 1)$ and $b = (0 : 0 : 1 : 0)$ respectively. We consider the planes $H' = \{z_4 = 0\}$, $H'' = \{z_3 = 0\}$ in $\mathbb{P}^3$, and we write

$$F_1 = X' \cap H' = \{f_1(z_1, z_2, z_3) = 0,\ z_4 = 0\}\ ,$$
$$F_2 = X'' \cap H'' = \{f_2(z_1, z_2, z_4) = 0,\ z_3 = 0\}\ ,$$

Date: June 24, 2003.

where f_1, f_2 are general homogeneous polynomials of degree m, n respectively. As we noted above, we have $X' = \langle F_1, a\rangle$, $X'' = \langle F_2, b\rangle$; hence, X is the surface of degree $m+n$ with equation

$$f_1(z_1, z_2, z_3)f_2(z_1, z_2, z_4) = 0 \ , \qquad (z_1 : z_2 : z_3 : z_4) \in \mathbb{P}^3.$$

We assume that $a \notin X''$ and $b \notin X'$, i.e. $f_1(0,0,1) \neq 0$ and $f_2(0,0,1) \neq 0$. Let

$$\pi_0 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \ , \qquad (z_1 : z_2 : z_3 : z_4) \mapsto (z_1 : z_2)$$

be the projection from the line $z_1 = z_2 = 0$. We further assume that F_1 and F_2 are smooth and that each fiber of $\pi_0|F_1$ and of $\pi_0|F_2$ has at least 3 distinct points. For example, if $m = n = 4$, this will be the case whenever $(0:0:1)$ does not lie on any of the bitangents or inflection tangent lines of $\{f_1 = 0\}$ or $\{f_2 = 0\}$.

We follow the deformation method of our paper [10]. Let $X_\infty = \{f_\infty = 0\}$ be a general surface of degree $m+n$ in $\mathbb{P}^3$, and let

$$X_t = \{f_1(z_1, z_2, z_3)f_2(z_1, z_2, z_4) + t\, f_\infty(z_1, z_2, z_3, z_4) = 0\} \qquad (t \in \mathbb{C}) \ .$$

We claim that X_t is hyperbolic for sufficiently small $t \neq 0$. Suppose on the contrary that X_{t_n} is not hyperbolic for a sequence $t_n \to 0$. Then by Brody's Theorem [1], there exists a sequence $\varphi_n : \mathbb{C} \to X_{t_n}$ of entire holomorphic curves such that

$$\|\varphi_n'(0)\| = \sup_{w \in \mathbb{C}} \|\varphi_n'(w)\| = 1, \qquad n = 1, 2, \dots$$

where the norm is measured with respect to the Fubini-Study metric in $\mathbb{P}^3$. Hence after passing to a subsequence, we can assume that φ_n converges to a nonconstant entire curve $\varphi : \mathbb{C} \to X$.

Since $X = X' \cup X''$, we may suppose without loss of generality that $\varphi(\mathbb{C}) \subset X'$. Consider the projection from a,

$$\pi_a : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2 , \qquad (z_1 : z_2 : z_3 : z_4) \mapsto (z_1 : z_2 : z_3).$$

Then $f_1 \circ \pi_a \circ \varphi = 0$; i.e, $\pi_a \circ \varphi(\mathbb{C}) \subset \{f_1 = 0\} \approx F_1$. Since F_1 is hyperbolic (it has genus ≥ 3), it follows that $\pi_a \circ \varphi$ is constant, and hence $\varphi(\mathbb{C})$ is contained in a projective line of the form

$$\langle p, a\rangle = \{(p_1 s_0 : p_2 s_0 : p_3 s_0 : s_1) \in \mathbb{P}^3 \mid (s_0 : s_1) \in \mathbb{P}^1\} \ ,$$

where $p = (p_1 : p_2 : p_3 : 0) \in F_1$. We notice that $(p_1, p_2) \neq (0,0)$ by the hypothesis that $f_1(0,0,1) \neq 0$.

Let $\Gamma = X' \cap X''$ denote the double curve of X, and suppose that $q \in \Gamma \cap \langle p, a\rangle = X'' \cap \langle p, a\rangle$. Recalling that $b \notin X'$, we see that q is of the form $q = (p_1 : p_2 : p_3 : s)$ where $f_2(p_1 : p_2 : s) = 0$. Thus we have a bijection

$$\Gamma \cap \langle p, a\rangle \xrightarrow{\approx} \pi_0^{-1}(p_1 : p_2) \cap F_2 \ , \qquad (p_1 : p_2 : p_3 : s) \mapsto (p_1 : p_2 : 0 : s) \ .$$

For general $x = (p_1 : p_2)$ the set $\pi_0^{-1}(x) \cap F_2$ contains $n \geq 4$ distinct points, and by our assumption, it contains at least 3 distinct points for all $x \in \mathbb{P}^1$. Hence $\Gamma \cap \langle p, a\rangle$ contains at least 3 distinct points for all $p \in F_1$, and contains n points for general $p \in F_1$.

Claim: $\varphi(\mathbb{C}) \subset \langle p, a\rangle \setminus (\Gamma \setminus X_\infty)$.

Proof of the claim: Suppose on the contrary that

$$\varphi(w_0) = (\zeta_1 : \zeta_2 : \zeta_3 : \zeta_4) \in \Gamma \setminus X_\infty$$

for some $w_0 \in \mathbb{C}$. Let Δ be a small disk about w_0 such that $\varphi(\bar{\Delta}) \cap X_\infty = \emptyset$. After shrinking Δ if necessary, we can lift the maps $\varphi_n|\bar{\Delta}$ via the projection $\pi : \mathbb{C}^4 \setminus \{0\} \to \mathbb{P}^3$ to maps $\widetilde{\varphi}_n : \bar{\Delta} \to \mathbb{C}^4$ such that

$$\widetilde{\varphi}_n \to \widetilde{\varphi}, \qquad \pi \circ \widetilde{\varphi} = \varphi|\bar{\Delta}.$$

(Simply choose j with $\zeta_j \neq 0$ and let $(\widetilde{\varphi}_n)_j \equiv 1$. Note that by our hypothesis that $a, b \notin \Gamma$, we can choose $j = 1$ or 2.)

Let n be sufficiently large so that $\varphi_n(\bar{\Delta})$ does not meet X_∞. Then $f_\infty \circ \widetilde{\varphi}_n$ does not vanish on $\bar{\Delta}$. Since $\varphi_n(\bar{\Delta}) \subset X_t$, it then follows from the equation for X_t that $f_2 \circ \widetilde{\varphi}_n$ cannot vanish on $\bar{\Delta}$ (where we write $f_2(z_1, z_2, z_3, z_4) = f_2(z_1, z_2, z_4)$). On the other hand, since $\varphi(w_0) \in X''$, we have $f_2 \circ \widetilde{\varphi}(w_0) = 0$. It then follows from Hurwitz's Theorem that $f_2 \circ \widetilde{\varphi} \equiv 0$, i.e. $\varphi(\Delta) \subset X''$. Then φ is constant since $\varphi(\Delta)$ lies in the finite set $X'' \cap \langle p, a\rangle$, a contradiction. This verifies the claim.

We now assume that, for all $p \in F_1$, the set $\langle p, a\rangle \cap (\Gamma \setminus X_\infty)$ contains at least 3 points, or in other words, the finite set $X_\infty \cap \Gamma$ does not contain 2 distinct points of $\langle p, a\rangle$, and does not contain any of the points $\Gamma \cap \langle p, a\rangle$ for the special values of p where $\Gamma \cap \langle p, a\rangle$ consists of only 3 points. Similarly, we make the same assumption for F_2. To show that this assumption holds for general X_∞, we consider the branched cover

$$\pi_\Gamma := \pi_0|\Gamma : \Gamma \to \mathbb{P}^1.$$

General fibers of π_Γ contain mn distinct points. It suffices to show that a general X_∞

i) does not contain 2 distinct points of any fiber of π_Γ (i.e. $\pi_0|(\Gamma \cap X_\infty)$ is injective), and
ii) does not contain any of the points of the special fibers with fewer than mn points.

Since the totality of points in (ii) is finite, (ii) certainly holds for general X_∞. It then suffices to show (i) for the nonspecial fibers. Since π_Γ is nonbranched at the points of the nonspecial fibers, these points are smooth points of Γ, and hence by Bertini's theorem, a general divisor X_∞ intersects Γ transversally at these points. Now suppose that $X_\infty = \{f_\infty = 0\}$ intersects Γ transversally and does not intersect the special fibers, and furthermore $\pi_0(\Gamma \cap X_\infty)$ has maximal cardinality among such X_∞. If (i) does not hold, then we can write $\Gamma \cap X_\infty = \{q^1, q^2, \ldots, q^{(m+n)mn}\}$, where $\pi_0(q^1) = \pi_0(q^2)$. Choose a divisor $Y = \{h = 0\}$ of degree $m + n$ containing the point q^1 but not q^2, and let $X_\infty^\varepsilon = \{f_\infty + \varepsilon h = 0\}$. For small ε, we let q_ε^j denote the point of $\Gamma \cap X_\infty^\varepsilon$ close to q^j. (These points are well defined and the maps $\varepsilon \mapsto q_\varepsilon^j$ are continuous for small ε, since by the transversality assumption, $f_\infty|\Gamma$ has only simple zeros.) Then $q_\varepsilon^1 = q^1$ and $q_\varepsilon^2 \neq q^2$ for small $\varepsilon \neq 0$. Hence for ε sufficiently small, $\pi_0(q_\varepsilon^2) \neq \pi_0(q^2) = \pi_0(q_\varepsilon^1)$ and $\#[\pi_0(\Gamma \cap X_\infty^\varepsilon)] > \#[\pi_0(\Gamma \cap X_\infty)]$, a contradiction.

Thus, $\langle p, a\rangle \cap (\Gamma \setminus X_\infty)$ contains at least 3 points, for all $p \in F_1$ (for general X_∞). Since $\varphi(\mathbb{C}) \subset \langle p, a\rangle \setminus (\Gamma \setminus X_\infty)$, φ is constant by Picard's Theorem, which is a contradiction. □

Remark: For an alternative construction of surfaces with hyperbolic deformations, we let $F = \{f = 0\}$ and $G = \{g = 0\}$ be two general plane curves of degrees $m \geq 4$ and $n \geq 2$, respectively. We suppose that the projective line $z_0 = 0$ meets F (G, respectively) transversally at m (n, respectively) distinct points $\{a_1, \ldots, a_m\}$ ($\{b_1, \ldots, b_n\}$, respectively). We then consider the following cones in $\mathbb{P}^4$ (with coordinates $(z_0 : \ldots : z_4)$) over these curves:

$$Y_1 := \langle F, u\rangle = \{f(z_0, z_1, z_2) = 0\} \qquad \text{and} \qquad Y_2 := \langle G, v\rangle = \{g(z_0, z_3, z_4) = 0\}\ ,$$

where the vertex sets are the skew projective lines

$$u := \{z_0 = z_1 = z_2 = 0\} \qquad \text{and} \qquad v := \{z_0 = z_3 = z_4 = 0\}\,.$$

We let $Y := Y_1 \cap Y_2$. Thus Y is an irreducible complete intersection surface in $\mathbb{P}^4$ of degree mn. It has $m+n$ singular points $\{A_1, \ldots, A_m\} = v \cap X$ of multiplicity n and $\{B_1, \ldots, B_n\} = u \cap Y$ of multiplicity m and no further singularities. Indeed, the hyperplane section $Y \cap H_\infty$, where $H_\infty := \{z_0 = 0\} \simeq \mathbb{P}^3$, is the union of mn distinct projective lines $l_{jk} := \langle A_j B_k \rangle$ $(j = 1, \ldots, m,\ k = 1, \ldots, n)$, n lines through each point A_j and m through each point B_k.

Then Y is birational to the direct product $F \times G$. Indeed, it is obtained by blowing up the mn points $c_{jk} := a_j \times b_k \in F \times G$ $(j = 1, \ldots, m,\ k = 1, \ldots, n)$, and then blowing down the proper transforms of m vertical generators $a_j \times G$ and n horizontal generators $F \times b_k$ to the singular points $A_j \in X$ and $B_k \in X$, $j = 1, \ldots, m$, $k = 1, \ldots, n$, respectively.

We let now $\pi : \mathbb{P}^4 \dashrightarrow H_\infty \simeq \mathbb{P}^3$ be a general projection with center $P_0 = (1:0:0:0) \notin Y \cup H_\infty$, and we let $Z := \pi(Y) \subset \mathbb{P}^3$ (with the coordinates $(z_1 : z_2 : z_3 : z_4)$). Then Z is given by the resultant $r := \mathrm{res}_{z_0}(f(z_0, z_1, z_2), g(z_0, z_3, z_4))$.

One can easily check in the same way as above that a general small deformation of Z is a hyperbolic surface in $\mathbb{P}^3$ of degree $mn \geq 8$.

The degenerate case $g = (z_0 - z_3)(z_0 - z_4)$ gives again the union of two cones $X = X' \cup X''$ as in the above theorem for the case $f_1 = f_2 = f$.

References

[1] Brody R. Compact manifolds and hyperbolicity. Trans. Amer. Math. Soc. 235 (1978), 213–219.

[2] Clemens H. Curves in generic hypersurfaces, Ann. Sci. Ecole Norm. Sup. 19 (1986), 629–636.

[3] Demailly J.-P., El Goul J. Hyperbolicity of generic surfaces of high degree in projective 3-space. Amer. J. Math. 122 (2000), 515–546.

[4] Duval J. Letter to J.-P. Demailly, October 30, 1999 (unpublished).

[5] El Goul J. Logarithmic jets and hyperbolicity (preprint 2001, arxiv.org/math.AG/0102128).

[6] Fujimoto H. A family of hyperbolic hypersurfaces in the complex projective space. The Chuang special issue. Complex Variables Theory Appl. 43 (2001), 273-283.

[7] McQuillan M. Holomorphic curves on hyperplane sections of 3-folds. Geom. Funct. Anal. 9 (1999), 370–392.

[8] Mori S., Mukai S. The uniruledness of the moduli space of curves of genus 11. Algebraic geometry (Tokyo/Kyoto, 1982), 334–353, Lecture Notes in Math., 1016, Springer, Berlin, 1983.

[9] Shiffman B., Zaidenberg M. Two classes of hyperbolic surfaces in $\mathbf{P}^3$. Intern. J. Math. 11 (2000), 65–101.

[10] Shiffman B., Zaidenberg M. Constructing low degree hyperbolic surfaces in $\mathbf{P}^3$. Special issue for S. S. Chern. Houston J. Math. 28 (2002), 377–388.

[11] Xu G. Subvarieties of general hypersurfaces in projective space. J. Differential Geom. 39 (1994), 139-172.

[12] Zaidenberg M. Stability of hyperbolic imbeddedness and construction of examples. Math. USSR Sbornik 63 (1989), 351-361.

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA
E-mail address: shiffman@math.jhu.edu

Université Grenoble I, Institut Fourier, UMR 5582 CNRS-UJF, BP 74, 38402 St. Martin d'Hères cédex, France
E-mail address: zaidenbe@ujf-grenoble.fr